

Using the smoothness of $p - 1$ for computing roots modulo p

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Abstract

We prove, without recourse to the Extended Riemann Hypothesis, that the projection modulo p of any *prefixed* polynomial with integer coefficients can be completely factored in deterministic polynomial time if $p - 1$ has a $(\ln p)^{O(1)}$ -smooth divisor exceeding $(p - 1)^{\frac{1}{2} + \delta}$ for some arbitrary small δ . We also address the issue of computing roots modulo p in deterministic time.

1 Introduction

Factoring polynomials over finite fields in deterministic polynomial time is a long-standing open problem of computational number theory. The most important results obtained, though partial, are now classic. Berlekamp [2] was the first to devise a general deterministic algorithm for this problem; its running time bound $p(d \ln p)^{O(1)}$, where p is the characteristic of the finite field and d the degree of the polynomial to be factored, can be seen as polynomial only if p is fixed. A better and so far best time bound $p^{\frac{1}{2}}(d \ln p)^{O(1)}$ is achieved by an algorithm of Shoup [15]. There are also algorithms with running time bound of the form $(d \ln p)^{O(1)}$, such as the Cantor-Zassenhaus algorithm [4] (actually dating back to Legendre), but these use randomness in an essential way. In this article we pursue an approach developed by von zur Gathen [6] and Rónyai [13], which consists in taking advantage of the multiplicative structure of $p - 1$. Shoup [16] refined the algorithms of von zur Gathen and Rónyai, improving the running time bound from

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$P^+(p-1)(d \ln p)^{O(1)}$ to $P^+(p-1)^{\frac{1}{2}}(d \ln p)^{O(1)}$, where $P^+(p-1)$ is the largest prime factor of $p-1$. These three algorithms are deterministic, however in their proofs of correctness the Extended Riemann Hypothesis must be assumed. We prove, without recourse to the ERH, that the projection modulo p of any *prefixed* polynomial with integer coefficients can be completely factored in deterministic polynomial time if $p-1$ has a $(\ln p)^{O(1)}$ -smooth divisor exceeding $(p-1)^{\frac{1}{2}+\delta}$ for some arbitrary small δ .

Theorem 1.1 *Let f be an irreducible polynomial of degree d in $\mathbb{Z}[X]$, with leading coefficient l , and \tilde{f} the polynomial of $\mathbb{Z}[Y]$ defined by $\tilde{f}(Y) = l^{d-1}f(\frac{Y}{l})$. Let h be the class number of $\mathbb{Q}(\theta)$, where θ is any complex root of \tilde{f} . Let p be a prime and q the least prime such that the q -smooth part S of $p-1$ is no less than $(p-1)^{\frac{1}{2}+\delta}$ for some $\delta > 0$. Then the complete factorization of f modulo p can be found in $O_{\theta,c,\delta,\epsilon}((q^{\frac{1}{2}} \ln q + \ln p) \ln^{2+\gamma+ch} p)$ deterministic time, where γ is any positive number and c, ϵ positive numbers satisfying $\frac{d}{c} + \epsilon < 2\delta$.*

Let us emphasize that the above big- O constant depends severely on θ as to include the time of some necessary precomputations in the number field $\mathbb{Q}(\theta)$. For a thorough exposition of constructive algebraic number theory we refer the reader to [11]. In a sense, the idea of fixing the polynomial f rather than the prime p is at the opposite of Berlekamp's algorithm. Actually, it has been already considered by Schoof [14] in the case when $f = X^2 - a$ for a an integer, and by Pila [9] when $f = X^s - 1$ for s a prime dividing $p-1$. Both authors gave unconditional algorithms that factor the corresponding polynomial f modulo p in respectively $O_a(\ln^{6+\epsilon} p)$ and $(\ln p)^{O_s(1)}$ deterministic time. We address more generally the issue of computing roots modulo p in the ensuing corollary. Our proofs are mainly based upon algebraic number theory, whereas Schoof's and Pila's rely on heavy machinery of algebraic geometry and this is of independent interest.

Corollary 1.2 *Let p be a prime and q the least prime such that the q -smooth part of $p-1$ is no less than $(p-1)^{\frac{1}{2}+\delta}$ for some $\delta > 0$. Let n be a positive integer. Suppose that the integer a is an n -th power residue modulo p . Then all the n -th roots of a modulo p can be computed in $O_{a,n,c,\delta,\epsilon}((q^{\frac{1}{2}} \ln q + \ln p) \ln^{2+\gamma+cH} p)$ deterministic time, where γ is any positive number, c, ϵ positive numbers satisfying $\frac{n}{c} + \epsilon < 2\delta$, and H the largest integer among the class numbers of the fields $\mathbb{Q}(\theta)$, θ running through the complex n -th roots of a .*

In the cases covered by Schoof or Pila and such that the corresponding integer H is equal to one the stated running time bound is slightly better for a sparse, but infinite set of primes p . It is worth noting that our technique combined with the result of Pila and an observation of Tsz-Wo Sze [17] leads for n an odd prime to a stronger theorem than corollary 1.2 (see last remark of section 5).

2 Notation

In all that follows, f is a *fixed*, irreducible polynomial of degree d in $\mathbb{Z}[X]$, with leading coefficient l , and \tilde{f} the corresponding monic, irreducible polynomial $l^{d-1}f(\frac{Y}{l})$ of degree d in $\mathbb{Z}[Y]$. The number field K is the extension of \mathbb{Q} by a complex root θ of the polynomial \tilde{f} . The class number of K is h , its ring of integers - \mathcal{O}_K . A *fixed*, integral basis $\omega = (\omega_1, \dots, \omega_d)$ of \mathcal{O}_K , as well as a *fixed*, finite set \mathcal{U} of generators of the group of units \mathcal{O}_K^* are given. The ideals of \mathcal{O}_K that we consider are always supposed to be nonzero. The norm $N(I)$ of an ideal in \mathcal{O}_K is the cardinality of \mathcal{O}_K/I . We let $\psi_K(x, y)$, respectively $\tilde{\psi}_K(x, y)$, be the number of ideals, respectively principal ideals, of \mathcal{O}_K with norm at most x that can be written as a product of prime ideals, respectively principal ideals, of \mathcal{O}_K with norm at most y .

The letter p denotes an odd prime number. For $g \in \mathbb{Z}_p[Y]$, by R_g we mean the quotient ring $\mathbb{Z}_p[Y]/(g)$ and by R_g^* its multiplicative group. If the commutative group G is a direct sum of two subgroups G_1, G_2 , and \mathcal{G} is a subset of G then the symbol $\langle \mathcal{G} \rangle_G$, respectively $\langle \mathcal{G} \rangle_{G_1}$, stands for the subgroup of G generated by \mathcal{G} , respectively the subgroup of G/G_2 generated by the cosets $gG_2, g \in \mathcal{G}$.

3 Auxiliary results

We will seek to compute the factorization of \tilde{f} modulo p ; it gives the factorization of f by a change of variable whenever p does not divide l . In general, we can assume that the prime p exceeds any given constant, since for small p factoring in $\mathbb{Z}_p[X]$ is "easy".

The algorithm of Fellows and Koblitz [5] for proving the primality of an integer n or also the deterministic version of Pollard's $p-1$ method [18] for factoring n , perform certain operations on a "small" subset \mathcal{B} of \mathbb{Z} that generates modulo n a "large" multiplicative semigroup \mathcal{S} . In fact, \mathcal{B} can be chosen there as the set of prime numbers not exceeding $\ln^2 n$; the semigroup \mathcal{S} has then at least $\psi(n, \ln^2 n)$ elements, where ψ is the de Bruijn function.

Here similarly, to factor \tilde{f} modulo p we will construct a small subset of $\mathbb{Z}_p[Y]/(\tilde{f})$ generating a large multiplicative semigroup. By the following lemma, the latter task amounts to exhibiting a suitable subset of \mathcal{O}_K , at least if p is sufficiently large.

Lemma 3.1 *Assume that p does not divide the index $[\mathcal{O}_K : \mathbb{Z}[\theta]]$. Then $\theta \mapsto Y$ induces an isomorphism $\kappa : \mathcal{O}_K/(p) \rightarrow \mathbb{Z}_p[Y]/(\tilde{f})$.*

Unlike \mathbb{Z} however, \mathcal{O}_K is not a unique factorization domain (unless $h = 1$). It is still a Dedekind domain and just as ψ measures the smoothness of integers in \mathbb{Z} , the function ψ_K measures the smoothness of ideals in \mathcal{O}_K . The next theorem generalizes in this sense a result of Canfield et al. [3].

Theorem 3.2 (Moree, Stewart) *There is an effective, positive constant $c_1 = c_1(K)$ such that for $x, y \geq 1$ and $u := \frac{\ln x}{\ln y} \geq 3$ we have*

$$\psi_K(x, y) \geq x \exp \left[-u \left\{ \ln(u \ln u) - 1 + \frac{\ln \ln u - 1}{\ln u} + c_1 \left(\frac{\ln \ln u}{\ln u} \right)^2 \right\} \right].$$

The generator of a principal ideal in \mathcal{O}_K is defined up to a multiplicative unit of \mathcal{O}_K , so working with principal ideals, rather than general ones, is pretty much like working with integers. That is why we give via ψ_K a lower bound for the function $\tilde{\psi}_K$ counting the number of "smooth" principal ideals.

Lemma 3.3 *There is an effective, positive constant $c_2 = c_2(K)$ such that $\tilde{\psi}_K(x, y) \geq \frac{1}{h} \psi_K(c_2 x, y^{\frac{1}{h}})$ for $y \geq c_2^{-h}$.*

Proof. Let I_1, \dots, I_h be a set of representatives for the class group of K whose norms are bounded above by the Minkowski bound M_K . We will prove that the lemma holds with $c_2 = M_K^{-1}$. Define $\psi'_K(x, y)$ as the number of principal ideals of \mathcal{O}_K with norm at most x that split as a product of prime ideals of \mathcal{O}_K with norm at most y .

Let J be an ideal counted by $\psi_K(M_K^{-1}x, y^{\frac{1}{h}})$. There exists a k , $1 \leq k \leq h$, such that JI_k is principal. Suppose that $y^{\frac{1}{h}} \geq M_K$, i.e. $y \geq M_K^h$. Then JI_k is counted by $\psi'_K(x, y^{\frac{1}{h}})$. Moreover, any ideal counted by $\psi'_K(x, y^{\frac{1}{h}})$ can be written in at most h ways as JI_k , where J is counted by $\psi_K(M_K^{-1}x, y^{\frac{1}{h}})$ and $1 \leq k \leq h$. Consequently, $\frac{1}{h} \psi_K(M_K^{-1}x, y^{\frac{1}{h}}) \leq \psi'_K(x, y^{\frac{1}{h}})$.

Assume that the principal ideal I of \mathcal{O}_K is a product of m prime ideals of \mathcal{O}_K with norm at most $y^{\frac{1}{h}}$. It is easy to show by induction on m that I is a product of principal ideals of \mathcal{O}_K with norm at most y . Just use the

fact that every product of at least h ideals of \mathcal{O}_K contains a principal factor. Therefore any ideal counted by $\psi'_K(x, y^{\frac{1}{h}})$ is also counted by $\tilde{\psi}_K(x, y)$, hence $\psi'_K(x, y^{\frac{1}{h}}) \leq \tilde{\psi}_K(x, y)$. \square

It becomes apparent that if we let \mathcal{B} be the union of \mathcal{U} and a set \mathcal{A} containing pairwise non-associate integers with small norm then it should generate modulo p a relatively large multiplicative semigroup \mathcal{S} . Nevertheless, three problems arise. Is \mathcal{S} indeed large though reduction modulo p ? Can the suitable set \mathcal{A} be small and easy to find? The ensuing theorem helps to answer these questions positively.

Theorem 3.4 (Fincke, Pohst) *There is an effective, positive constant $c_3 = c_3(K, \omega)$ such that for any $\eta \in \mathcal{O}_K \setminus \{0\}$ there exists $\tilde{\eta} \in \mathcal{O}_K$ generating the same ideal as η and whose coordinates a_i in the basis ω satisfy $|a_i| \leq c_3 N((\eta))^{\frac{1}{d}}$.*

Proof. Combine the equations (3.5b), chapter 5, and (4.3f), chapter 6 of [11]. \square

We now summarize rigorously the above informal discussion. Actually, we show more: for any g dividing f modulo p , a set \mathcal{B}_g derived from \mathcal{B} generates a large multiplicative semigroup in R_g .

Lemma 3.5 *Suppose that the polynomial g of degree d' divides \tilde{f} modulo p . Let p and κ be as in lemma 3.1. Let π and π_g be the projections $\mathcal{O}_K \rightarrow \mathcal{O}_K/(p)$ and $\mathbb{Z}_p[Y]/(\tilde{f}) \rightarrow \mathbb{Z}_p[Y]/(g)$ respectively. Fix $c > 0$ and define $\mathcal{A} = \{a_1\omega_1 + \dots + a_d\omega_d : a_i \in \mathbb{Z}, |a_i| \leq c_3 \ln^{\frac{ch}{d}} p, 1 \leq i \leq d\}$, $\mathcal{S} = \{v \cdot \alpha_1 \cdot \dots \cdot \alpha_m : v \in \mathcal{O}_K^*, m \in \mathbb{N}, \alpha_i \in \mathcal{A}, 1 \leq i \leq m\}$. Then $\#\pi_g \kappa \pi(\mathcal{S}) > p^{d' - \frac{d}{c} - \epsilon}$ for any $\epsilon > 0$ and $p \geq p_0$, $p_0 = p_0(c, c_1, c_2, c_3, \epsilon)$.*

Proof. Let $\mathcal{T} = \mathcal{S} \cap \{a_1\omega_1 + \dots + a_d\omega_d : a_i \in \mathbb{Z}, |a_i| \leq \frac{p}{2}, 1 \leq i \leq d\}$. It is sufficient to prove that the desired inequality holds with \mathcal{S} replaced by \mathcal{T} . We invoke theorem 3.4 to get $\#\mathcal{T} \geq \tilde{\psi}_K((\frac{p}{2c_3})^d, \ln^{ch} p)$. By lemma 3.3, if p is large enough the latter expression is no less than $\frac{1}{h} \psi_K(c_2(\frac{p}{2c_3})^d, \ln^c p)$. This in turn is greater than $p^{d - \frac{d}{c} - \epsilon}$ for any $\epsilon > 0$ and sufficiently large p , by theorem 3.2. Thus $\#\mathcal{T} > p^{d - \frac{d}{c} - \epsilon}$ if p exceeds some constant p_0 depending upon c, c_1, c_2, c_3 and ϵ . Assume that it does. As $p > 2$, we have $\#\pi(\mathcal{T}) = \#\mathcal{T}$. Furthermore, κ is an isomorphism, hence $\#\kappa\pi(\mathcal{T}) = \#\pi(\mathcal{T})$. Finally, π_g is a surjective homomorphism, so the preimage under π_g of any element of $\mathbb{Z}_p[Y]/(g)$ has $\#\ker \pi_g = p^{d-d'}$ elements. It follows that $p^{d-d'} \cdot \#\pi_g \kappa \pi(\mathcal{T}) \geq \#\kappa\pi(\mathcal{T}) = \#\mathcal{T} > p^{d - \frac{d}{c} - \epsilon}$. Therefore $\#\pi_g \kappa \pi(\mathcal{T}) > p^{d' - \frac{d}{c} - \epsilon}$. \square

Assume that g is a product of at least two distinct, degree e irreducible factors of f modulo p . Either the set \mathcal{B}_g mentioned above is not contained in $R_g^* \cup \{0\}$, or $\mathcal{G} := \{b^{\frac{p^e-1}{p-1}} : b \in \mathcal{B}_g \setminus \{0\}\}$ generates a large subgroup of $\{a \in R_g^* : a^{p-1} = 1\}$ and thus should not be cyclic. The latter case is dealt with an extension of the Pohlig-Hellman algorithm [10] for computing discrete logarithms.

Theorem 3.6 *Let g be a polynomial of degree d' in $\mathbb{Z}_p[Y]$ and G the group $\{a \in R_g^* : a^{p-1} = 1\}$. Write $G = G_1 \oplus G_2$, $(\#G_1, \#G_2) = 1$. Suppose that we are given g , $\#G_1$ and a subset \mathcal{G} of G such that $\langle \mathcal{G} \rangle_{G_1}$ is not cyclic. Then we can find a nontrivial divisor of g in $O_{d'}(\#\mathcal{G} \cdot (q^{\frac{1}{2}} \ln q + \ln p) \ln^{2+\gamma} p)$ deterministic time, where q is largest prime factor of $\#G_1$ and γ any positive number.*

Proof. The deterministic Pollard-Strassen [12] algorithm can be used to find the complete factorization of the q -smooth part of $p-1$ in the stated time. The rest of the proof is based on obvious modifications of the proofs of corollary 4.4, theorem 6.6 and on remark 4.3 from [18]. \square

4 Proof of theorem 1.1

If $p \leq \max \left(l, [\mathcal{O}_K : \mathbb{Z}[\theta]], p_0, (1 - p_0^{-1})^{\frac{-d\delta-1}{2\delta-\frac{d}{e}-\epsilon}} \right)$, where p_0 is the constant from lemma 3.5, then we can find efficiently the complete factorization of \tilde{f} using the deterministic Berlekamp algorithm for example. Now assume that the reverse inequality holds. We first compute the squarefree, distinct-degree factorization of \tilde{f} modulo p , that is the products t_e , $e \in \mathbb{N}$, of all distinct, degree e irreducible divisors of \tilde{f} modulo p . Fix e ; the complete factorization of t_e will be found by using the following inductive procedure. Let g be a factor of t_e of degree d' , say $d' = ke$. Suppose that $k \geq 2$. We show below how to split g nontrivially. Keep the notation of lemma 3.5. Define $\mathcal{B}_g = \pi_g \kappa \pi(\mathcal{U} \cup \mathcal{A})$. We can assume that $\mathcal{B}_g \subset R_g^* \cup \{0\}$; in the contrary case (b, g) is a nontrivial divisor of g for some $b \in \mathcal{B}_g$. Let $\mathcal{F} = \mathcal{B}_g \setminus \{0\}$. With G as in theorem 3.6, let \mathcal{G} be the image of \mathcal{F} under the homomorphism $\sigma : R_g^* \rightarrow G$ raising every element to the power $\frac{p^e-1}{p-1}$. Write G as $G_1 \oplus G_2$ with $\#G_1 = S^k$ - this condition uniquely determines G_1 and G_2 . We apply the algorithm from theorem 3.6 to check whether the group $\langle \mathcal{G} \rangle_{G_1}$ is cyclic. Suppose that it is, for otherwise we would obtain a nontrivial factor of g . Then the order of $\langle \mathcal{G} \rangle_{G_1}$ divides $p-1$. We will estimate this order from below to obtain a contradiction. We have $\#\langle \mathcal{G} \rangle_{G_1} = \# \frac{\langle \mathcal{G} \rangle_G}{\langle \mathcal{G} \rangle_{G \cap G_2}}$.

The kernel of σ has $\left(\frac{p^e-1}{p-1}\right)^k$ elements, hence $\#\langle\mathcal{G}\rangle_G \geq \left(\frac{p-1}{p^e-1}\right)^k \cdot \#\langle\mathcal{F}\rangle_{R_g^*}$. We appeal to lemma 3.5 to deduce that $\#\langle\mathcal{F}\rangle_{R_g^*} > p^{d'-\frac{d}{c}-\epsilon} - 1$. Since $S \geq (p-1)^{\frac{1}{2}+\delta}$, it follows that $\#\langle\mathcal{G}\rangle_G \cap G_2 \leq \#G_2 \leq (p-1)^{\frac{k}{2}-k\delta}$. Therefore $\#\langle\mathcal{G}\rangle_{G_1} > (p-1)^{\frac{k}{2}+k\delta} \cdot \frac{p^{d'-\frac{d}{c}-\epsilon}-1}{(p^e-1)^k}$. The right hand side of this inequality is easily seen to be no less than $p-1$, which gives the desired contradiction. This means that a nontrivial divisor of g had to be found at some stage. Once we have completely factored \tilde{f} in $\mathbb{Z}_p[Y]$, we get the complete factorization of f in $\mathbb{Z}_p[X]$ by the change of variable $Y = lX$. Obviously, the most time-consuming part of the described procedure is testing whether $\langle\mathcal{G}\rangle_{G_1}$ is cyclic. The stated running time follows from theorem 3.6, as $\#\mathcal{G} = O_d(\ln^{ch} p)$.

5 Concluding remarks

Theorem 3.6 can be applied to get a short proof of Shoup's result from [16], which states that under the ERH there is an algorithm that completely factors *any* degree d polynomial f in $\mathbb{Z}_p[X]$ in $P^+(p-1)^{\frac{1}{2}}(d \ln p)^{O(1)}$ deterministic time. It suffices to iterate the following procedure. Let g be a reducible factor of f . Adopt the notation of theorem 3.6. As in von zur Gathen's algorithm, either we find directly a nontrivial divisor of g , or compute an element a of $G \setminus \mathbb{Z}_p$ whose order is the power of some prime s (cf. [1]). Then we find an s -th power nonresidue b modulo p . Finally we use the algorithm from theorem 3.6 with $G_1 = G$ and $\mathcal{G} = \{a, b\}$ to find a nontrivial divisor of g . All the required steps can be done in the stated time.

Instead of completely factoring we could be simply interested in splitting the polynomial f modulo p . To this end, the running time bound obtained in theorem 1.1 could be in some cases largely improved. For example, if the degree of f is odd then it would be sufficient to take the integer q therein as the least prime such that the q -smooth part of $p-1$ is no less than $(p-1)^{\frac{1}{3}+\delta}$ for some $\delta > 0$.

As another example, consider the polynomial $f = X^s - a$, where s is an odd prime number and the integer a is not an s -th power. Suppose that f splits modulo p into distinct linear factors, or equivalently a is an s -th power residue modulo p and s divides $p-1$. In order to split f modulo p within the time bound of theorem 1.1 it is then enough to choose q as the least prime such that the q -smooth part of $p-1$ is no less than $(p-1)^{\frac{1}{s}+\delta}$ for some $\delta > 0$. The point is that a nontrivial factor of f modulo p leads to an s -th root of a modulo p (cf. [17]). The remaining s -th roots of a and

hence the complete factorization of f modulo p can be found by computing a primitive s -th root of unity modulo p . This in turn can be done using Pila's algorithm [9].

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